

ON QUANTITATIVE SCHUR AND DUNFORD-PETTIS PROPERTIES

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ABSTRACT. We show that the dual to any subspace of $c_0(\Gamma)$ has the strongest possible quantitative version of the Schur property. Further, we establish relationship between the quantitative Schur property and quantitative versions of the Dunford-Pettis property. Finally, we apply these results to show, in particular, that any subspace of the space of compact operators on ℓ_p ($1 < p < \infty$) with Dunford-Pettis property satisfies automatically both its quantitative versions.

1. THE MAIN RESULT

A Banach space X is said to have the *Schur property* if any weakly null sequence in X converges to zero in norm. Equivalently, X has the Schur property if every weakly Cauchy sequence is norm Cauchy. The classical example of a space with the Schur property is the space ℓ_1 of all absolutely summable sequences.

A quantitative version of the Schur property was introduced and studied in [5]. Let us recall the definition. If (x_k) is a bounded sequence in a Banach space X , we set (following [5])

$$\text{ca}(x_k) = \inf_{n \in \mathbb{N}} \text{diam}\{x_k : k \geq n\}$$

and

$$\delta(x_k) = \sup_{x^* \in B_{X^*}} \inf_{n \in \mathbb{N}} \text{diam}\{x^*(x_k) : k \geq n\}.$$

Then the quantity $\text{ca}(\cdot)$ measures how far the sequence is from being norm Cauchy, while the quantity $\delta(\cdot)$ measures how far it is from being weakly Cauchy. It is easy to check that the quantity $\delta(x_k)$ can be alternatively described as the diameter of the set of all weak* cluster points of (x_k) in X^{**} . Following again [5], a Banach space X is said to have the *C-Schur property* (where $C \geq 0$) if

$$(1.1) \quad \text{ca}(x_k) \leq C\delta(x_k)$$

for any bounded sequence (x_k) in X . Since obviously $\delta(x_k) \leq \text{ca}(x_k)$ for any bounded sequence (x_k) , necessarily $C \geq 1$ (unless X is the trivial space). Moreover, if X has the C -Schur property for some $C \geq 1$, it easily follows that X has the Schur property. Indeed, if (x_k) is weakly Cauchy in X , then $\delta(x_k) = 0$, and thus $\text{ca}(x_k) = 0$. The space constructed in [5, Example 1.4] serves as an example of a Banach space with the Schur property without the C -Schur property for any $C > 0$. On the other hand, $\ell_1(\Gamma)$ possesses the 1-Schur property (see [5, Theorem 1.3]). Our main result is the following generalization of the quoted theorem.

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Theorem 1.1. *Let X be a subspace of $c_0(\Gamma)$. Then X^* has the 1-Schur property.*

Let us now proceed to the proof of the main result.

We will need some lemmas. The first one establishes a special property of the norm on $c_0(\Gamma)$ and its subspaces.

Lemma 1.2. *Let X be a subspace of $c_0(\Gamma)$. Then for any $x^* \in X^*$ and any sequence (x_n^*) in X^* which weak* converges to 0 we have*

$$\limsup \|x_n^* + x^*\| = \|x^*\| + \limsup \|x_n^*\|.$$

Proof. Let us first suppose that X is separable. It is obvious that for any $x \in X$ and any weakly null sequence (x_n) in X we have

$$\limsup \|x_n + x\| = \max(\|x\|, \limsup \|x_n\|).$$

The assertion then follows from [6, Theorem 2.6] (applied for $p = \infty$).

The general case follows by a separable reduction argument. Suppose that $x^* \in X^*$ and that (x_n^*) is a weak* null sequence in X^* . Let us consider the countable set

$$A = \{x^*\} \cup \{x_n^* : n \in \mathbb{N}\} \cup \{x_n^* + x^* : n \in \mathbb{N}\}.$$

We can find a separable subspace $Y \subset X$ such that for each $y^* \in A$ we have $\|y^*\| = \|y^*|_Y\|$. Then the assertion follows immediately from the separable case. \square

The next one is a stronger variant of [1, Lemma 1.7] or [6, Lemma 2.3] for the special case of subspaces of $c_0(\Gamma)$.

Lemma 1.3. *Let X be a subspace of $c_0(\Gamma)$ and (x_n^*) be sequence in X^* weak* converging to x^* . Then for any finite dimensional subspace $F \subset X^*$ we have*

$$\liminf \text{dist}(x_n^*, F) \geq \liminf \|x_n^*\| - \|x^*\|.$$

Proof. Let $c > \liminf \text{dist}(x_n^*, F)$ be arbitrary. By passing to a subsequence we may assume that $\text{dist}(x_n^*, F) < c$ for each $n \in \mathbb{N}$. We can thus find a sequence (y_n^*) in F such that $\|x_n^* - y_n^*\| < c$ for each $n \in \mathbb{N}$. Since the sequence (x_n^*) is bounded, the sequence (y_n^*) is bounded as well. Therefore we can, up to passing to a subsequence, suppose that the sequence (y_n^*) converges in norm to some $y^* \in F$. Then

$$\begin{aligned} c &\geq \limsup \|x_n^* - y_n^*\| = \limsup \|x_n^* - y^*\| = \limsup \|(x_n^* - x^*) + (x^* - y^*)\| \\ &= \limsup \|x_n^* - x^*\| + \|x^* - y^*\| \geq \limsup \|x_n^*\| - \|x^*\| + \|x^* - y^*\| \\ &\geq \liminf \|x_n^*\| - \|x^*\|. \end{aligned}$$

The first equality follows from the fact that the sequence (y_n^*) converges to y^* in the norm, the third one follows from Lemma 1.2. The remaining steps are trivial.

This completes the proof. \square

The next lemma is a refinement of constructions from [5, Lemma 2.1] and [1, Theorem 1.1]. During its proof we will use the following notation: if $x \in c_0(\Gamma)$ or $x \in \ell_1(\Gamma)$ and $A \subset \Gamma$, then $x|_A$ denotes an element defined as

$$(x|_A)(\gamma) = \begin{cases} x(\gamma), & \gamma \in A, \\ 0, & \gamma \in \Gamma \setminus A. \end{cases}$$

Lemma 1.4. *Let X be a subspace of $c_0(\Gamma)$, $c > 0$ and (y_n) be a sequence in $\ell_1(\Gamma) = c_0(\Gamma)^*$ such that*

- (y_n) weak* converges to 0 in $\ell_1(\Gamma)$,
- $\|y_n|_X\| > c$ for each $n \in \mathbb{N}$.

Then for any $\eta > 0$ there is a subsequence (y_{n_k}) such that each weak cluster point of $(y_{n_k}|_X)$ in X^{***} has norm at least $c - \eta$.*

Proof. For $n \in \mathbb{N}$ set $\varphi_n = y_n|_X$. Let $\varepsilon \in (0, \frac{c}{6})$ be arbitrary. Without loss of generality, we may assume that $\varepsilon < 1$. We select strictly positive numbers (ε_k) such that $\sum_{k=1}^{\infty} \varepsilon_k < \varepsilon$.

We inductively construct elements $x_k \in X$, indices $n_1 < n_2 < \dots$ and finite sets $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \dots \subset \Gamma$ such that, for each $k \in \mathbb{N}$,

- (a) $\|x_k\| \leq 1$, $x_k|_{\Gamma_{k-1}} = 0$ and $\|x_k|_{\Gamma \setminus \Gamma_k}\| < \varepsilon_k$,
- (b) $|\varphi_{n_k}(x_k)| > c - \varepsilon$ and $|\varphi_{n_k}(\sum_{i=1}^{k-1} x_i)| \leq \varepsilon \cdot \|\sum_{i=1}^{k-1} x_i\|$,
- (c) if we denote $y_{n_k}^1 = y_{n_k}|_{\Gamma_k}$ and $y_{n_k}^2 = y_{n_k}|_{\Gamma \setminus \Gamma_k}$, then $\|y_{n_k}^2\| < \varepsilon_k$.

In the first step, we set $\Gamma_0 = \emptyset$ and $n_1 = 1$. Since $\|\varphi_{n_1}\| > c$, there is $x_1 \in B_X$ with $|\varphi_{n_1}(x_1)| > c$. Let us choose a finite set $\Gamma_1 \subset \Gamma$ satisfying

$$\|x_1|_{\Gamma \setminus \Gamma_1}\| < \varepsilon_1 \quad \text{and} \quad \|y_{n_1}|_{\Gamma \setminus \Gamma_1}\| < \varepsilon_1.$$

Since the second requirement in (b) is vacuous, the first step is finished.

Assume now that we have found indices $n_1 < \dots < n_k$, finite sets $\emptyset = \Gamma_0 \subset \dots \subset \Gamma_k$ and elements x_1, \dots, x_k satisfying (a), (b) and (c). We define an operator $R_k : X \rightarrow c_0(\Gamma)$ as

$$R_k x = x|_{\Gamma_k}, \quad x \in X.$$

Then $\text{Ker } R_k$ is of finite codimension, and thus $F_k = (\text{Ker } R_k)^\perp$ is a finite dimensional space in X^* . Let $m \in \mathbb{N}$ be chosen such that, for each $n \geq m$,

- $|\varphi_n(\sum_{i=1}^{k-1} x_i)| \leq \varepsilon \cdot \|\sum_{i=1}^{k-1} x_i\|$, and
- $\text{dist}(\varphi_n, F_k) > c - \varepsilon$.

(The first requirement can be fulfilled due to the fact that (φ_n) converges weak* to 0, and the second one due to Lemma 1.3.) Let $n_{k+1} = m$ and

$$x_{k+1} \in (F_k)_\perp = \text{Ker } R_k$$

be chosen such that $\|x_{k+1}\| \leq 1$ and

$$\varphi_{n_{k+1}}(x_{k+1}) > c - \varepsilon$$

(we use the fact that $X^*/F_k = ((F_k)_\perp)^*$). We find a finite set $\Gamma_{k+1} \supset \Gamma_k$ satisfying

$$\|x_{k+1}|_{\Gamma \setminus \Gamma_{k+1}}\| < \varepsilon_{k+1} \quad \text{and} \quad \|y_{n_{k+1}}|_{\Gamma \setminus \Gamma_{k+1}}\| < \varepsilon_{k+1}.$$

This finishes the construction.

For $J \in \mathbb{N}$, let

$$u_J = \sum_{i=1}^J x_i.$$

It follows from (a) that, for each $k \in \mathbb{N}$ and $J > k$, we have

$$(1.2) \quad \left\| \sum_{i=1}^k x_i \right\| < 1 + \varepsilon, \quad \left\| \sum_{i=1}^{k-1} x_i \right\| < 1 + \varepsilon, \quad \left\| \sum_{i=k+1}^J x_i \right\| < 1 + \varepsilon.$$

Indeed, for $k \in \mathbb{N}$ and $\gamma \in \Gamma_k \setminus \Gamma_{k-1}$, we have from (a)

$$|x_j(\gamma)| \leq \begin{cases} \varepsilon_j, & j < k, \\ 1, & j = k, \\ 0, & j > k, \end{cases} \quad j \in \mathbb{N}.$$

Further, x_k is bounded by ε_k on $\Gamma \setminus \bigcup_{k=1}^{\infty} \Gamma_k$ by (a). This observations verify (1.2).

For each $k \in \mathbb{N}$, we set

$$\varphi_{n_k}^1 = y_{n_k}^1|_X \quad \text{and} \quad \varphi_{n_k}^2 = y_{n_k}^2|_X.$$

For a fixed index $k \in \mathbb{N}$ and arbitrary $J > k$, we need to estimate

$$(1.3) \quad |\varphi_{n_k}(u_J)| = \left| \varphi_{n_k} \left(\sum_{i=1}^{k-1} x_i \right) + \varphi_{n_k}(x_k) + \varphi_{n_k} \left(\sum_{i=k+1}^J x_i \right) \right|.$$

The condition (b) and (1.2) ensures that

$$(1.4) \quad \left| \varphi_{n_k} \left(\sum_{i=1}^{k-1} x_i \right) \right| \leq \varepsilon \cdot \left\| \sum_{i=1}^{k-1} x_i \right\| < \varepsilon(1 + \varepsilon).$$

From (b) we also have

$$(1.5) \quad |\varphi_{n_k}(x_k)| > c - \varepsilon.$$

Finally, (a) and (c) give

$$(1.6) \quad \begin{aligned} \left| \varphi_{n_k} \left(\sum_{i=k+1}^J x_i \right) \right| &= \left| (\varphi_{n_k}^1 + \varphi_{n_k}^2) \left(\sum_{i=k+1}^J x_i \right) \right| \\ &= \left| y_{n_k}^2 \left(\sum_{i=k+1}^J x_i \right) \right| \leq \varepsilon_k \cdot \left\| \sum_{i=k+1}^J x_i \right\| \\ &< \varepsilon_k(1 + \varepsilon). \end{aligned}$$

Using (1.4)–(1.6) in (1.3), we get

$$(1.7) \quad \begin{aligned} |\varphi_{n_k}(u_J)| &\geq c - \varepsilon - \varepsilon(1 + \varepsilon) - \varepsilon_k(1 + \varepsilon) \\ &\geq c - \varepsilon(3 + 2\varepsilon) \geq c - 5\varepsilon. \end{aligned}$$

It follows from (1.7) that, for $z_J = (1 + \varepsilon)^{-1}u_J$, we have $z_J \in B_X$ by (1.2) and

$$|\varphi_{n_k}(z_J)| > (1 + \varepsilon)^{-1}(c - 5\varepsilon), \quad k \in \mathbb{N}, J > k.$$

Let $z^{**} \in B_{X^{**}}$ be a weak* cluster point of (z_J) . Then

$$(1.8) \quad |\varphi_{n_k}(z^{**})| \geq (1 + \varepsilon)^{-1}(c - 5\varepsilon), \quad k \in \mathbb{N}.$$

It follow that each weak* cluster point of (φ_{n_k}) has norm at least $(1 + \varepsilon)^{-1}(c - 5\varepsilon)$.

This completes the proof, as given $\eta > 0$, we can in the beginning choose ε such that

$$(1 + \varepsilon)^{-1}(c - 5\varepsilon) > c - \eta.$$

□

Now we are ready to prove the theorem:

Proof of Theorem 1.1. Let X be a subspace of $c_0(\Gamma)$ and (x_n^*) be a sequence in X^* bounded by a constant M . We consider arbitrary $0 < c < \text{ca}(x_n^*)$. We extract subsequences (a_n) and (b_n) from (x_n^*) such that

$$(1.9) \quad c < \|a_n - b_n\|, \quad n \in \mathbb{N}.$$

We denote $\varphi_n = a_n - b_n$, $n \in \mathbb{N}$. We extend a_n to $A_n \in \ell_1(\Gamma)$ and φ_n to $z_n \in \ell_1(\Gamma)$ with preservation of the norm and set $B_n = A_n - z_n$. Then B_n is an extension of b_n (not necessarily preserving the norm). By passing to a subsequence if necessary, assume that (A_n) converges pointwise (and hence weak* in $\ell_1(\Gamma)$) to some $A \in \ell_1(\Gamma)$ and (B_n) converges pointwise to some $B \in \ell_1(\Gamma)$. (This is possible due to the fact that any sequence in $\ell_1(\Gamma)$ can be viewed as a sequence in $\ell_1(\Gamma')$ for a countable $\Gamma' \subset \Gamma$.) Then (z_n) weak* converges to $A - B$. Set $y_n = z_n - A + B$ for $n \in \mathbb{N}$. Then (y_n) weak* converges to 0 and $\|y_n|_X\| > c - \|(A - B)|_X\|$ for each $n \in \mathbb{N}$.

Let $\varepsilon > 0$ be arbitrary. By Lemma 1.4, there is a subsequence (y_{n_k}) such that each weak* cluster point of $(y_{n_k}|_X)$ in X^{***} has norm at least

$$c - \|(A - B)|_X\| - \varepsilon.$$

Let a be a weak* cluster point of (a_{n_k}) in X^{***} . Let (a_τ) be a subnet of (a_{n_k}) weak* converging to a . Let b be a weak* cluster point of the net (b_τ) . Then a and b are weak* cluster points of (x_n^*) in X^{***} .

Obviously $a|_X = A|_X$ and $b|_X = B|_X$ and, moreover, $a - b - (a - b)|_X = a - b - (A - B)|_X$ is a weak* cluster point of $(y_{n_k}|_X)$ in X^{***} . Thus

$$\|a - b - (a - b)|_X\| \geq c - \|(A - B)|_X\| - \varepsilon.$$

Further, let $F \in (\ell_\infty(\Gamma))^* = c_0(\Gamma)^{***}$ be an extension of $a - b$ with preserving the norm. Then

$$\begin{aligned} \|a - b\| &= \|F\| = \|F|_{c_0(\Gamma)}\| + \|F - F|_{c_0(\Gamma)}\| \geq \|F|_X\| + \|(F - F|_{c_0(\Gamma)})|_{X^{**}}\| \\ &= \|(A - B)|_X\| + \|a - b - (a - b)|_X\| \\ &\geq \|(A - B)|_X\| + c - \|(A - B)|_X\| - \varepsilon \\ &= c - \varepsilon. \end{aligned}$$

(Let us remark that, for a Banach space Y and $G \in Y^{***}$, we denote by $G|_Y$ the respective element of Y^* canonically embedded into Y^{***} .) It follows that $\delta(x_k^*) \geq c - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\delta(x_k^*) \geq c$. Hence $\text{ca}(x_k^*) \leq \delta(x_k^*)$ and the proof is completed. \square

2. QUANTITATIVE SCHUR PROPERTY AND QUANTITATIVE DUNFORD-PETTIS PROPERTY

It is well known that the Schur property is closely related to the Dunford-Pettis property. Recall that a Banach space X is said to have the *Dunford-Pettis property* if for any Banach space Y every weakly compact operator $T : X \rightarrow Y$ is completely continuous. Let us further recall that T is *weakly compact* if the image by T of the unit ball of X is relatively weakly compact in Y , and that T is *completely continuous* if it maps weakly convergent sequences to norm convergent ones, or, equivalently, if it maps weakly Cauchy sequence to norm Cauchy (hence norm convergent) ones.

Obviously, any Banach space with the Schur property has the Dunford-Pettis property. Further, any Banach space whose dual has the Schur property enjoys the Dunford-Pettis property as well.

Quantitative variants of the Dunford-Pettis property were studied in [3] where two strengthenings of the Dunford-Pettis property in a quantitative way were introduced (*direct quantitative Dunford-Pettis property* and *dual quantitative Dunford-Pettis property*, see [3, Definition 5.6]). Section 6 of [3] shows several relations between the Schur property and the two variants of the quantitative Dunford-Pettis properties. In this section we focus on the relationship of the quantitative Schur property and quantitative versions of the Dunford-Pettis property.

The unexplained notation and notions in this section are taken from [3].

More specifically, the quantities $\text{ca}_{\rho^*}(\cdot)$ and $\text{ca}_{\rho}(\cdot)$ measure how far the given sequence is from being Cauchy in the Mackey topology of X^* or the restriction to X of the Mackey topology of X^{**} , respectively. The quantity $\tilde{\delta}(\cdot)$ is defined by taking infimum of $\delta(\cdot)$ over all subsequences. Similarly for $\tilde{\text{ca}}(\cdot)$, $\tilde{\text{ca}}_{\rho^*}(\cdot)$ and $\tilde{\text{ca}}_{\rho}(\cdot)$. These quantities are defined and described in detail in [3, Section 2.3].

Further, $\widehat{\text{d}}(\cdot, \cdot)$ is the non-symmetrized Hausdorff distance, $\chi(\cdot)$ denotes the Hausdorff measure of norm non-compactness, $\omega(\cdot)$ and $\text{wk}_X(\cdot)$ are measures of weak non-compactness; see [3, Section 2.5]. To apply a measures of (weak) non-compactness to an operator means to apply it to the image of the unit ball (see [3, Section 2.6]).

Finally, the quantity $\text{cc}(\cdot)$ measures how far the given operator is from being completely continuous, i.e. if $T : X \rightarrow Y$ is an operator, then

$$\text{cc}(T) = \sup\{\text{ca}(Tx_k) : (x_k) \text{ is a weakly Cauchy sequence in } B_X\},$$

see [3, Section 2.4].

It is obvious that a Banach space X with the Schur property possesses also the direct quantitative Dunford-Pettis property (see [3, Proposition 6.2]). If we assume that X has a C -Schur property, we get the following result.

Theorem 2.1. *Let X be a Banach space with the C -Schur property where $C > 0$.*

- (i) *It holds $\text{ca}_{\rho}(x_n) \leq C\delta(x_n)$ for any bounded sequence (x_n) in X . In particular, X has both the direct and the dual quantitative Dunford-Pettis properties.*
- (ii) *The space X satisfies the following stronger version of the dual quantitative Dunford-Pettis property: If $A \subset X$ is a bounded set, then*

$$(2.1) \quad \text{wk}_X(A) \leq \omega(A) = \chi(A) \leq 2C \text{wk}_X(A).$$

Proof. The inequality in assertion (i) follows from the fact that $\text{ca}_{\rho}(x_n) \leq \text{ca}(x_n)$ for any bounded sequence (x_n) in X (this is an immediate consequence of definitions). Thus X satisfies condition (iv) of [3, Theorem 5.5], i.e., X possesses the dual quantitative Dunford-Pettis property. Further, from [3, Proposition 6.2] we know that X has the direct quantitative Dunford-Pettis property.

(ii) First we notice that (2.1) is indeed a stronger version of the dual quantitative Dunford-Pettis property. Indeed, using [3, diagramm (3.1) and formula (2.6)] one can deduce from (2.1) the validity of condition (i) of [3, Theorem 5.5].

For the proof of (2.1), let A be a bounded set in X . If (x_k) in X is a bounded sequence, by taking consecutively infima in (1.1) over all subsequences we obtain

$$(2.2) \quad \tilde{\text{ca}}(x_k) \leq C\tilde{\delta}(x_k).$$

By [4, Theorem 1],

$$(2.3) \quad \tilde{\delta}(x_k) \leq 2\widehat{\text{d}}(\text{clust}_{X^{**}}(x_k), X)$$

for any bounded sequence (x_k) in an arbitrary Banach space, and thus (2.3) together with (2.2) yield

$$(2.4) \quad \tilde{ca}(x_k) \leq 2C\widehat{d}(\text{clust}_{X^{**}}(x_k), X).$$

Since obviously (cf. [3, inequalities (2.2)])

$$\chi(A) \leq \sup\{\tilde{ca}(x_k) : (x_k) \text{ is a sequence in } A\},$$

(2.4) yields

$$(2.5) \quad \chi(A) \leq 2C \text{wk}_X(A).$$

Since X has the C -Schur property, it has the Schur property, and thus any weakly compact subset of X is norm compact. Hence

$$(2.6) \quad \chi(A) = \omega(A).$$

A consecutive use of [3, inequality (2.4)], (2.6), and (2.5) gives

$$\text{wk}_X(A) \leq \omega(A) = \chi(A) \leq 2C \text{wk}_X(A),$$

which is the inequality (2.1). \square

If the dual X^* of a Banach space X possesses the Schur property, then we have by [3, Theorem 6.3] that X has the dual quantitative Dunford-Pettis property and, moreover, for any Banach space Y and an operator $T : X \rightarrow Y$ the following inequalities hold:

$$(2.7) \quad \text{wk}_Y(T) \leq \omega(T) \leq \chi(T) \leq \text{cc}(T) \leq 2\omega(T^*) = 2\chi(T^*) \leq 4\chi(T).$$

Thus the quantities $\chi(T)$, $\text{cc}(T)$, $\chi(T^*)$ and $\omega(T^*)$ are equivalent in this case. However, the quantities $\omega(T)$ and $\text{wk}_Y(T)$ need not be in this case equivalent with the others, i.e., X need not have the direct quantitative Dunford-Pettis property, see [3, Example 10.1]. However, if we assume that X^* has a quantitative version of the Schur property, we obtain that, for an operator T with domain X , that the compactness (both norm and weak) of T and its adjoint are quantitatively equivalent to the complete continuity of T .

Theorem 2.2. *Let X be a Banach space such that X^* have the C -Schur property for some $C \geq 0$. If Y is a Banach space and $T : X \rightarrow Y$ is a bounded linear operator, we have*

$$(2.8) \quad \begin{aligned} \text{wk}_Y(T) &\leq \omega(T) \leq \chi(T) \leq \text{cc}(T) \\ &\leq 2\omega(T^*) = 2\chi(T^*) \leq 4C \text{wk}_{X^*}(T^*) \leq 8C \text{wk}_Y(T). \end{aligned}$$

In particular, X has both the direct and the dual quantitative Dunford-Pettis properties.

Proof. The first five inequalities are contained in [3, Theorem 6.3(i)]. By Theorem 2.1 we get the sixth inequality. The last inequality follows from [3, equation (2.8)]. Further, X^* has both the direct and dual quantitative Dunford-Pettis property by Theorem 2.1(i). Hence X itself possesses both the direct and dual quantitative Dunford-Pettis property by [3, Theorem 5.7]. \square

If we combine the previous theorem with Theorem 1.1, we get immediately.

Corollary 2.3. *Let X be a subspace of $c_0(\Gamma)$. Then X has both the direct and dual quantitative Dunford-Pettis properties. Moreover, the inequalities (2.8) are satisfied with $C = 1$.*

In case $X = c_0(\Gamma)$ [3, Theorem 8.2] yields even stronger inequalities (with $C = 1/2$). The proof of this case is done by a different method.

We continue by a characterization of spaces whose dual has the quantitative Schur property. It is well known that the dual space X^* of a Banach space X has the Schur property if and only if X has the Dunford-Pettis property and contains no copy of ℓ_1 (see [2, Theorem 3]). The following theorem quantifies this assertion.

Theorem 2.4. *Let X be a Banach space. Then X^* has the quantitative Schur property if and only if X has the direct quantitative Dunford-Pettis property and contains no copy of ℓ_1 .*

Proof. Suppose that X^* has the quantitative Schur property. Then X contains no copy of ℓ_1 . Indeed, if X contains an isomorphic copy of ℓ_1 , by [7, Proposition 3.3] the dual space X^* contains an isomorphic copy of $C(\{0, 1\}^{\mathbb{N}})^*$, hence also an isomorphic copy of $C([0, 1])^*$. The space $C([0, 1])^*$ fails the Schur property as it contains a copy of $L^1(0, 1)$. Thus X^* fails the Schur property as well. Further, X has the direct quantitative Dunford-Pettis property by Theorem 2.1.

For the proof of the converse implication we need the following consequence of Rosenthal's ℓ_1 -theorem.

Lemma 2.5. *Let X be a Banach space not containing an isomorphic copy of ℓ_1 . Then any bounded sequence (x_n^*) in X^* satisfies $\text{ca}(x_n^*) \leq 3 \text{ca}_{\rho^*}(x_n^*)$.*

Proof. If (x_n^*) is norm-Cauchy, then the inequality is obvious. So, suppose that $\text{ca}(x_n^*) > 0$ and fix any $c \in (0, \text{ca}(x_n^*))$. Then there is a sequence of natural numbers $l_n < m_n < l_{n+1}$, $n \in \mathbb{N}$, and a sequence (x_n) in B_X such that $|(x_{l_n}^* - x_{m_n}^*)(x_n)| > c$ for every $n \in \mathbb{N}$. By Rosenthal's ℓ_1 -theorem, there is a weakly Cauchy subsequence of (x_n) . Let us assume, without loss of generality, that $l_n = 2n - 1$ and $m_n = 2n$ for every $n \in \mathbb{N}$ and that (x_n) is weakly Cauchy.

Since, for every $k \in \mathbb{N}$, the singleton $\{x_k\}$ is a weakly compact set in B_X , there is some $n_k > k$ such that $|(x_{2n_k-1}^* - x_{2n_k}^*)(x_k)| < \text{ca}_{\rho^*}(x_n^*) + \frac{1}{k}$. Using this estimate and the fact that $\{\frac{x_{n_k} - x_k}{2} : k \in \mathbb{N}\}$ is a relatively weakly compact subset of B_X , we can write

$$\begin{aligned} c &\leq \limsup |(x_{2n_k-1}^* - x_{2n_k}^*)(x_{n_k})| \\ &\leq 2 \limsup |(x_{2n_k-1}^* - x_{2n_k}^*)(2^{-1}(x_{n_k} - x_k))| + \limsup |(x_{2n_k-1}^* - x_{2n_k}^*)(x_k)| \\ &\leq 2 \text{ca}_{\rho^*}(x_n^*) + \limsup (\text{ca}_{\rho^*}(x_n^*) + \frac{1}{k}) = 3 \text{ca}_{\rho^*}(x_n^*). \end{aligned}$$

This completes the proof. \square

Suppose now that X has the direct Dunford-Pettis property. Then there exists $C > 0$ such that

$$\text{ca}_{\rho^*}(x_n^*) \leq C \delta(x_n^*)$$

for any bounded sequence (x_n^*) in X^* (see [3, Theorem 5.4(iv)]). By Lemma 2.5,

$$\text{ca}(x_n^*) \leq 3 \text{ca}_{\rho^*}(x_n^*) \leq 3C \delta(x_n^*)$$

for any bounded sequence (x_n^*) in X^* . Hence X^* has the $3C$ -Schur property. \square

3. SUBSPACES OF THE SPACE OF COMPACT OPERATORS

The space $K(\ell_2)$ of all compact operators on the Hilbert space ℓ_2 can be viewed as a non-commutative version of c_0 and its dual $N(\ell_2)$, the space of all nuclear operators on ℓ_2 equipped with the nuclear norm, can be viewed as a non-commutative version of ℓ_1 . The non-commutative versions share many properties of the commutative ones, but Schur property and Dunford-Pettis property are essentially commutative.

Indeed, $N(\ell_2)$ does not have the Schur property and, moreover, $K(\ell_2)$ does not enjoy the Dunford-Pettis property. It is witnessed by the following easy example. Let (e_n) denote the standard basis in ℓ_2 . Consider the operators $T_n(x) = \langle x, e_1 \rangle e_n$, $x \in \ell_2$, and $S_n(x) = \langle x, e_n \rangle e_1$. These operators are rank-one operators, thus they are nuclear and hence compact. Moreover, both sequences converge weakly to 0 both in $K(\ell_2)$ and $N(\ell_2)$. The Schur property of $N(\ell_2)$ can be disproved by observing that $\|S_n\| = \|T_n\| = \|e_1\| \|e_n\| = 1$. Moreover, the failure of the Dunford-Pettis property of $K(\ell_2)$ follows by the fact that $\text{Tr}(S_n T_n) = 1$.

This easy observation was strengthened in [8], where the authors show that a subspace of $K(\ell_p)$, the space of compact operators on ℓ_p enjoys the Dunford-Pettis property if and only if it is isomorphic to a subspace of c_0 (i.e., only in the “commutative case”). Theorem 1.1 enables us to complement and strengthen their result to show that such a space has automatically a quantitative Dunford-Pettis property.

More precisely, we prove the following:

Theorem 3.1. *Let X be a subspace of the space $K(\ell_p)$ of compact operators on ℓ_p where $1 < p < \infty$. Then the following assertions are equivalent:*

- (i) *X has the Dunford-Pettis property.*
- (ii) *X^* has the Schur property.*
- (iii) *X is isomorphic to a subspace of c_0 . Moreover, in this case, there is for each $\varepsilon > 0$ an isomorphic embedding $T : X \rightarrow c_0$ such that $\|T\| \|T^{-1}\| < 4 + \varepsilon$.*
- (iv) *X^* has the 4-Schur property.*
- (v) *For each Banach space Y and each bounded linear operator $T : X \rightarrow Y$, the inequalities (2.8) hold with $C = 4$.*
- (vi) *The space X has both the dual and the direct quantitative Dunford-Pettis properties.*

Proof. The implication (ii) \Rightarrow (i) is well known (see [2, Theorem 3]).

(i) \Rightarrow (iii) If X has the Dunford-Pettis property, it is embeddable into c_0 by [8, Theorem 1]. Moreover, the constant of embedding can be explicitly computed from [8, Lemma 1 and 2]. Indeed, the embedding $T : X \rightarrow c_0$ is constructed as the composition $\psi \circ \phi_A$, where ϕ_A is provided by [8, Lemma 1] and ψ is provided by [8, Lemma 2]. The operator ψ satisfies $\|\psi\| \|\psi^{-1}\| \leq 4$ by [8, p. 420]. Further, ϕ_A satisfies $\|\phi_A\| \|\phi_A^{-1}\| \leq 3$ (see the computation in [8, p. 418]), but it can be easily modified to be an almost isometry. Indeed, if we replace in [8, formula (3) on p. 420] the number $\frac{1}{4}$ by $\frac{\varepsilon}{2}$, then we will obtain $\|\phi_A\| \|\phi_A^{-1}\| \leq \frac{1+\varepsilon}{1-\varepsilon}$. This completes the proof.

The implication (iii) \Rightarrow (iv) follows from Theorem 1.1. Indeed, let $T : X \rightarrow c_0$ be an embedding with $\|T\| = 1$ and $\|T^{-1}\| \leq 4 + \varepsilon$. Let (x_n^*) be a bounded sequence in X^* . Then $((T^*)^{-1} x_n^*)$ is a bounded sequence in $(T(X))^*$ satisfying $\delta((T^*)^{-1} x_n^*) \leq (4 + \varepsilon) \delta(x_n^*)$. By Theorem 1.1 we get $\text{ca}((T^*)^{-1} x_n^*) \leq 2(4 +$

$\varepsilon)\delta(x_n^*)$, hence $\text{ca}(x_n^*) \leq 2(4 + \varepsilon)\delta(x_n^*)$ as well. Since $\varepsilon > 0$ is arbitrary, the proof is finished.

The implications (iv) \Rightarrow (v) and (v) \Rightarrow (vi) follows from Theorem 2.2. Finally, the implications (vi) \Rightarrow (i) and (iv) \Rightarrow (ii) are trivial. \square

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